Computable Family of Σ_a^{-1}-Sets without Friedberg Numberings

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Abstract. We construct a computable family of Σ_a^{-1}–sets without Σ_α^{-1}–computable Friedberg numberings, where a is a notation for a constructive ordinal. We also construct a family of Σ_a^{-1}–sets without Σ_a^{-1}–computable Friedberg numberings which have Σ_b^{-1}–computable Friedberg numbering, where b is a notation for the successor of $|a|_{\mathcal{O}}$.

Key words: Ershov hierarchy, computable numberings

1 Main Results

One of the main questions of the theory of computable numberings is the study of extremal elements in Roger semilattice. First results in this area were obtained by Friedberg. He proved the existence of a single–valued computable numbering of the family of all computably enumerable sets. Later on it was shown in [1] that there exists a Σ_2^{-1}–computable family of Σ_2^{-1}–sets without Σ_2^{-1}–computable Friedberg numbering.

In this paper we generalize this result to all constructive ordinals.

Now we give the original Ershov's definition of the class Σ_a^{-1}, where a is a notation for a constructive ordinal.

Definition 1 ([2]). *Let $P(x, y)$ be a computable partial ordering on ω. A uniformly c.e. sequence $\{R_x\}$ of c.e. sets is called P–sequence if for all x and y the condition $x \leq_P y$ implies $R_x \subseteq R_y$.*

Hereinafter we will use Kleene ordinal notation system $(\mathcal{O}, <_o)$. For every $a \in \mathcal{O}$, $|a|_o$ is the ordinal α whose $\mathcal{O}$–notation is a. We also define a parity function $e(x)$: for all $a \in \mathcal{O}$, $e(a) = 1$ if $|a|_{\mathcal{O}}$ is even and $e(a) = 0$ if $|a|_{\mathcal{O}}$ is odd.

Definition 2. *For every $a \in \mathcal{O}$, we define the operation S_a that takes a–sequences $\{R_x\}_{x<_o a}$ to subsets of ω:*

$$S_a(R) = \{z | \exists x <_o a(z \in R_x \;\&\; e(x) \neq e(a) \;\&\; \forall y <_o x(z \notin R_y))\}$$

For $a \in \mathcal{O}$, the class Σ_a^{-1} is defined as the class of all sets $S_a(R)$, where $R = \{R_x\}_{x<_o a}$ is any a–sequence of c.e. sets.

In this paper we will use another definition of Σ_a^{-1}–sets.

Definition 3. *For all $a \in O$, a set A is a Σ_a^{-1}–set if there exist total computable function $f(x,s)$ and partial computable function $g(x,s)$ such that for all $x \in \omega$ the following conditions are satisfied:*

1. $A(x) = \lim_s f(x,s),\ f(x,0) = 0$
2. $g(x,s) \downarrow \rightarrow g(x,s+1) \downarrow \leq_o g(x,s) <_o a$
3. $f(x,s) \neq f(x,s+1) \rightarrow g(x,s+1) \downarrow \neq g(x,s)$.

This definition is not well-known and our next step is proving of equivalence of given definitions.

Lemma 1. *Definitions 2 and 3 are equivalent.*

Proof. "$\Leftarrow$". Take a computable function $f(x,s)$ and a partial computable function $g(x,s)$ as in the definition. Define an a–sequence of c.e. sets $\{R_b\}$ as follows. For all $b <_o a$, let

$$R_b = \bigcup_{c <_o b} R_c \cup \{x | (\exists s)(\exists t \leq_o b)(f(x,s) = |e(b) - e(a)| \ \&\ g(x,s) = t)\},$$

$$A = \{x | \exists b <_o a (x \in R_b \ \&\ e(b) \neq e(a) \ \&\ \forall c <_o b (x \notin R_c))\}.$$

"$\Rightarrow$". We have an a–sequence $\{R_b\}$ of c.e. sets. Construct a total computable function $f(x,s)$ and a partial computable function $g(x,s)$. Since $\{R_b\}$ is a uniformly c.e. sequence, there exists a uniformly computable sequence $\{R_b^s\}_{s\in\omega}$ of computable sets $R_b^0 \subseteq R_b^1 \subseteq R_b^2 \subseteq \ldots$ such that for all $b <_o a$ holds $R_b = \bigcup_{s\in\omega} R_b^s$.

Define an auxiliary function $h(x,s)$ as follows:

1. $h(x,0) = 0$;
2. (a) if $x \in R^s_{h(x,s)}$ or $h(x,s) = b$, where b is a notation for the predecessor of $|a|_o$, then we let $h(x,s+1) = 0$;
   (b) if $x \notin R^s_{h(x,s)}$ then $h(x,s+1) = \mu_{c <_o a}(h(x,s) <_o c)$.

Now we can describe constructions for $f(x,s)$ and $g(x,s)$:

1. $f(x,0) = 0$, $g(x,0)$ is undefined;
2. (a) if $x \in R^s_{h(x,s)}$ then $f(x,s+1) = |e(h(x,s)) - e(a)|$, $g(x,s+1) = h(x,s)$;
   (b) if $x \notin R^s_{h(x,s)}$ then $f(x,s+1) = f(x,s)$, $g(x,s+1) = g(x,s)$.

It is easy to see that functions $f(x,s)$ and $g(x,s)$ satisfy all the properties of Definition 3. Lemma is complete.

Definition 4 ([3]). *A numbering ν is said to be Σ_a^{-1}–computable if there exist a total computable function $f(n,x,s)$ and a partial computable function $g(n,x,s)$ such that for all $n,x \in \omega$ holds*

1. $\nu(n,x) = \lim_s f(n,x,s),\ f(n,x,0) = 0$;
2. $g(n,x,s) \downarrow \rightarrow g(n,x,s+1) \downarrow \leq_o g(n,x,s) <_o a$;
3. $f(n,x,s) \neq f(n,x,s+1) \rightarrow g(n,x,s+1) \downarrow \neq g(n,x,s)$.

Now we can formulate the main result of this paper.

Theorem 1. *For any $a \in \mathcal{O}$, there exists a Σ_a^{-1}–computable family without Σ_a^{-1}–computable Friedberg numberings.*

In the remaining part of the paper, we prove this theorem and its corollary.

2 Proof of Theorem

Firstly, we prove a useful technical result:

Lemma 2. *There exists an effective list of all Σ_a^{-1}–computable numberings.*

Proof. Let Ξ_a^{-1} be an m–universal set for the family of all Σ_a^{-1}–sets (the existence of such sets is shown, for example, in [4]). Then for all $n \in \omega$, $\kappa_n^{-1}(\Xi_a^{-1})$ is a Σ_a^{-1}–set, where κ is Kleene's numbering of all partial computable functions, and for each Σ_a^{-1}–set A there exists a n with the property $A = \kappa_n^{-1}(\Xi_a^{-1})$. Moreover, for any Σ_a^{-1}–computable numbering ν there exists a total computable function f with property $\nu = \kappa_{f(n)}^{-1}(\Xi_a^{-1})$ and for each total computable function f, $\kappa_{f(n)}^{-1}(\Xi_a^{-1})$ is a Σ_a^{-1}–computable numbering. Let $\phi_e(n)$ be some universal function for class of all partial computable functions. Define $\mu_e(n)$ as follows:

$$\mu_e(n) = \kappa_{\phi_e(n)}^{-1}(\Xi_a^{-1}).$$

If $\phi_e(n)$ is undefined, then $\kappa_{\phi_e(n)}(x)$ is undefined for all $x \in \omega$ and $\mu_e(n) = \emptyset$. It is easy to see, that $\mu_e(n)$ will be effective list of all Σ_a^{-1}–computable numberings. Lemma is complete.

Proceed to the proof of theorem. Fix an effective list of all Σ_a^{-1}–computable numberings μ_e of Σ_a^{-1}–sets and build a Σ_a^{-1}–computable numbering ν of Σ_a^{-1}–sets. It follows from the definitions that to define a numbering ν, we should construct a total computable function $f(n, x, s)$ and a partial computable function $g(n, x, s)$. Since μ_e is Σ_a^{-1}–computable, there are a list of total computable functions $\phi_e(n, x, s)$ and a list of partial computable functions $\psi_e(n, x, s)$ such that each pair $\langle \phi_e, \psi_e \rangle$ defines a numbering μ_e.

Now we can define f and g.

For all $n, x \in \omega$, let $f(n, x, 0) = 0$ and $g(n, x, 0) \uparrow$.

For each $e \in \omega$ we are waiting for a stage s' and distinct indices i and j such that $\phi_e(i, 2e, s') = 1$ and $\phi_e(j, 2e+1, s') = 1$. While s' does not occur, for each stage s $(0 < s < s')$ we let $f(2e, 2e, s) = f(2e+1, 2e+1, s) = 1$ and $g(2e, 2e, s) = g(2e+1, 2e+1, s) = 1$.

For all $e' \neq 2e, 2e+1$ we let $f(e', 2e, s) = f(e', 2e+1, s) = 0$ and $g(e', 2e, s) \uparrow$, $g(e', 2e+1, s) \uparrow$.

When the required s' occurs, for all $s \geq s'$ we put $f(2e, 2e, s) = f(2e+1, 2e+1, s) = 0$, $g(2e, 2e, s) = g(2e+1, 2e+1, s) = 0$, and for all $e' \neq 2e, 2e+1$ we put $f(e', 2e, s) = \overline{sg}[\phi_e(i, 2e, s)]$, $f(e', 2e+1, s) = \overline{sg}[\phi_e(j, 2e+1, s)]$, $g(e', 2e, s) = \psi_e(i, 2e, s)$ $g(e', 2e+1, s) = \psi_e(j, 2e+1, s)$.

Comments to the construction.

1. At the very beginning, we enumerate $2e$ in $\nu(2e)$ and $2e+1$ in $\nu(2e+1)$ (But if the stage s' we are waiting for equals 1, we do not enumerate elements in $\nu(2e)$ and in $\nu(2e+1)$; these sets remain empty forever).
2. We are waiting for a stage s' and distinct indices i, j, such that $2e \in \mu_{e,s'}(i)$ and $2e+1 \in \mu_{e,s'}(j)$.

3. We remove elements $2e$ $2e+1$ from sets $\nu(2e)$ and $\nu(2e+1)$ respectively.
4. For all $s \geq s'$ and all $e' \neq 2e, 2e+1$, if an element $2e$ will be extracted from set $\mu_{e,s}(i)$ then we add the element $2e$ to the set $\nu(e')$. If $2e \in \mu_{e,s}(i)$ holds again then $2e$ should be extracted from $\nu(e')$. The same actions should be performed for the element $2e+1$ and the set $\mu_{e,s}(j)$.

Let S be a family of Σ_a^{-1}–sets enumerated by ν. Suppose that μ_e enumerates a family T. If there are no stage s' and indices i, j satisfying the conditions described in the construction then T does not contain distinct sets containing elements $2e$ and $2e+1$, although, there are such sets in S. By this, $S \neq T$. If such a stage s' and indices i, j exist then the following two cases may occur:

1. At least one of the conditions $2e \in \mu_e(i)$ and $2e+1 \in \mu_e(j)$ is satisfied. Suppose that $2e \in \mu_e(i)$. Then in S there are no sets containing $2e$, but there are such sets in T, for example, $\mu_e(i)$.
2. Both $2e \notin \mu_e(i)$ and $2e+1 \notin \mu_e(j)$ are true. Then only one set $\nu(2e) = \nu(2e+1)$ in S does not contain $\{2e, 2e+1\}$. It follows from $S = T$ that $\mu_e(i) = \mu_e(j)$ for distinct indices i and j. Thus, μ_e cannot be a Friedberg numbering.

Some modifications of this construction enable us to produce a family of Σ_a^{-1}–sets with more complicated structure.

Corollary 1. *There is a computable family of Σ_a^{-1}–sets without Σ_a^{-1}–computable Friedberg numbering which has a Σ_b^{-1}–computable Friedberg numbering, where b is a notation for the successor of $|a|_{\mathcal{O}}$.*

Again, to produce a numbering ν, we construct a total computable function $f(n, x, s)$ and a partial computable function $g(n, x, s)$.

Describe a construction.

For all $n, x \in \omega$, we let $f(n, x, 0) = 0$ and $g(n, x, 0) \uparrow$.

For each $e \in \omega$, let $f(3e+2, 3e+2, s) = 1$ and $g(3e+2, 3e+2, s) = 0$, for each $s > 0$.

We are waiting for a stage s' and distinct indices i and j such that $\phi_e(i, 3e, s') = 1$ and $\phi_e(j, 3e+1, s') = 1$. While such an s' did not occur, for each stage s $(0 < s < s')$ we put $f(3e, 3e, s) = f(3e+1, 3e+1, s) = 1$ and $g(3e, 3e, s) = g(3e+1, 3e+1, s) = 1$.

For all $e' \neq 3e, 3e+1, 3n+2$, where $n \in \omega$, we let $f(e', 3e, s) = f(e', 3e+1, s) = 0$ and $g(e', 3e, s) \uparrow, g(e', 3e+1, s) \uparrow$.

After the required s' occurs, we define for all $s \geq s'$: $f(3e, 3e, s) = f(3e+1, 3e+1, s) = 0$ and $g(3e, 3e, s) = g(3e+1, 3e+1, s) = 0$.

For all $e' \neq 3e, 3e+1, 3n+2$, where $n \in \omega$, we let $f(e', 3e, s) = \overline{sg}[\phi_e(i, 3e, s)]$, $f(e', 3e+1, s) = \overline{sg}[\phi_e(j, 3e+1, s)]$, and $g(e', 3e, s) = \psi_e(i, 3e, s)$ $g(e', 3e+1, s) = \psi_e(j, 3e+1, s)$.

In this construction, we execute the same steps (with elements $3e$ and $3e+1$ instead of elements $2e$ and $2e+1$, and the set $\nu(3e+2)$ always contains only one element $3e+2$). The same argument shows that this is a family of Σ_a^{-1}–sets without Σ_a^{-1}–computable Friedberg numbering.

Let b be a notation for the successor of $|a|_{\mathcal{O}}$. Construct a Σ_b^{-1}–computable numbering η of this family (again, we build a total computable function $\varphi(n,x,s)$ and a partial computable function $\psi(n,x,s)$).

For all $n, x \in \omega$, let $\varphi(n,x,0) = 0$ and $\psi(n,x,0) \uparrow$.

Further in construction, we act for each $e \in \omega$.

For all $e', s \in \omega$, let $\varphi(2e, e', s) = f(3e, e', s)$ and $\psi(2e, e', s) = g(3e, e', s)$.

We are waiting for a stage s' such that $f(3e, 3e, s') = f(3e+1, 3e+1, s') = 0$. While such s' did not occur, for each stage s ($0 < s < s'$) and for all $e' \in \omega$ we put $\varphi(2e+1, e', s) = f(3e+1, e', s)$ and $\psi(2e+1, e', s) = c$, where c is a notation for the successor of $|g(3e+1, e', s)|_{\mathcal{O}}$.

After the required s' occurs, for all $s \geq s'$ and for all $e' \neq 3x + 2$, where x is minimal natural number which was not used in previous steps we define $\varphi(2e + 1, e', s) = 0$, $\psi(2e + 1, e', s) = 0$, $\varphi(2e + 1, 3x + 2, s) = 1$, and $\psi(2e + 1, 3x + 2, s) = 0$.

Comments to the construction.

1. The sets $\eta(2e)$ and $\nu(3e)$ are the same, for all e.
2. While $3e + 1$ and $3e + 2$ remain in $\nu(3e + 1)$ and $\nu(3e + 2)$ respectively, we construct the set $\eta(2e + 1)$ so that it would be equal to $\nu(3e + 1)$. If at some moment $3e + 1$ and $3e + 2$ leave the sets $\nu(3e + 1)$ and $\nu(3e + 2)$ then these sets become equal. In this case we construct $\eta(2e + 1)$ as a new set $\nu(3x + 2) = \{3x + 2\}$.

It follows from the construction that the numbering η is Σ_b^{-1}–computable and Friedberg.

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