Families with Infinite Rogers Semilattices in Ershov Hierarchy

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Abstract. We show, that any computable family of Σ_a^{-1}–sets has positive and infinitely many computable numberings on some higher level of Ershov hierarchy.

Key words: Ershov hierarchy, Rogers semilattices, computable numberings

1 Introduction

Study the cardinality and the structure of Roger semilattices of families of sets in different hierarchies is one of the main questions in numberings theory. Here we concentrate our interest on Roger semilattices in Ershov hierarchy. The results in this area were obtained by various researchers, such as Badaev, Goncharov, Lempp, Talasbaeva and others. We will especially mention two papers, that were, in some way, a motivation of this work. First result ([1]) is proved by Goncharov, Lempp, Solomon. They showed, that the family of all sets from finite level of Ershov hierarchy has a Friedberg numbering. The second result([2]) is by Badaev and Lempp and it is about a decomposition of the Rogers semilattice of a family of d.c.e. sets.

There is well known result([5]), that the Rogers semilattice of any family $\{A, B\}$, with $A \subset B$, A, B - computable enumerable sets, is infinite. Here we construct some generalization of this fact onto Ershov hierarchy. After we prove, that any computable family of sets from some level of Ershov hierarchy has a positive numbering on some higher level. The connection of these results gives our main result, that any computable family of sets from some level of Ershov hierarchy has infinitely many nonequivalent computable numberings on some higher level.

2 Main Definitions

Hereinafter we use Kleene ordinal notation system $(\mathcal{O}, <_o)$. For every $a \in \mathcal{O}$, $|a|_\mathcal{O}$ is the ordinal α whose $\mathcal{O}$-notation is a. We also define a parity function $e(x)$: for all $a \in \mathcal{O}$, $e(a) = 1$ if $|a|_\mathcal{O}$ is even and $e(a) = 0$ if $|a|_\mathcal{O}$ is odd.

We call a set from level $|a|_\mathcal{O}$ of Ershov hierarchy([3]) a Σ_a^{-1}-set.

Definition 1. *For all $a \in O$, a set A is a Σ_a^{-1}-set if there exist total computable function $f(x,s)$ and partial computable function $g(x,s)$ such that for all $x \in \omega$ the following conditions are satisfied:*

1. $A(x) = \lim_s f(x,s),\ f(x,0) = 0$
2. $g(x,s) \downarrow \rightarrow g(x,s+1) \downarrow \leq_{\mathcal{O}} g(x,s) <_{\mathcal{O}} a$
3. $f(x,s) \neq f(x,s+1) \rightarrow g(x,s+1) \downarrow \neq g(x,s)$.

A pair $\langle f,g \rangle$ is called a Σ_a^{-1}-approximation of a set A.

We call a set A a Π_a^{-1}-set, if its complement $\overline{A}$ is Σ_a^{-1}-set. Due to previous definition it is possible to say, that for such set $f(x,0) = 1$.

Now we give definition of Σ_a^{-1}-computable numbering(in sense of [4]).

Definition 2. *A numbering ν is said to be Σ_a^{-1}-computable if there exist a total computable function $f(n,x,s)$ and a partial computable function $g(n,x,s)$ such that for all $n,x \in \omega$ holds*

1. $\nu_n(x) = \lim_s f(n,x,s),\ f(n,x,0) = 0$;
2. $g(n,x,s) \downarrow \rightarrow g(n,x,s+1) \downarrow \leq_{\mathcal{O}} g(n,x,s) <_{\mathcal{O}} a$;
3. $f(n,x,s) \neq f(n,x,s+1) \rightarrow g(n,x,s+1) \downarrow \neq g(n,x,s)$.

A numbering ν is called *positive* if a set $\{< x,y > | \nu_x = \nu_y\}$ is computable enumerable and *Friedberg* if it enumerates the family of sets without repetitions.

Definition 3. *A Σ_a^{-1}-approximation of a set A is called fair parity Σ_a^{-1}- approximation, if*

1. $f(x,s) = 0$, *then* $e(g(x,s)) = e(a)$ *or* $g(x,s)$ *is undefined*
2. $f(x,s) = 1$, *then* $e(g(x,s)) = 1 - e(a)$

Now we prove a useful technical result:

Lemma 1. *if a set has Σ_a^{-1}-approximation then it has fair parity Σ_a^{-1}- approximation.*

Let pair $\langle \phi, \psi \rangle$ be Σ_a^{-1}-approximation. We construct a pair $\langle f,g \rangle$.

$f(x,s) = \phi(x,s)$ for all $x,s \in \omega$.

1. if $\psi(x,s)$ is undefined then $g(x,s)$ is also undefined
2. if $\psi(x,s)$ is defined then
   - $\psi(x,s) = b$, where $a = 2^b$($|a|_{\mathcal{O}}$ is successor of $|b|_{\mathcal{O}}$)
   -- if $\phi(x,s) = 0$ then $g(x,s)$ is undefined (there are no changes have been made yet)
   -- if $\phi(x,s) = 1$ then $g(x,s) = \psi(x,s)$
   - if $e(\psi(x,s)) \neq e(a)$:
   -- $\phi(x,s) = 0 \Rightarrow g(x,s) = c$, where $|c|_{\mathcal{O}}$ is the successor of $|\psi(x,s)|_{\mathcal{O}}$
   -- $\phi(x,s) = 1 \Rightarrow g(x,s) = \psi(x,s)$
   - if $e(\psi(x,s)) = e(a)$:
   -- $\phi(x,s) = 1 \Rightarrow g(x,s) = c$, where $|c|_{\mathcal{O}}$ is the successor of $|\psi(x,s)|_{\mathcal{O}}$
   -- $\phi(x,s) = 0 \Rightarrow g(x,s) = \psi(x,s)$

Hereinafter "$+_{\mathcal{O}}$" is a partial computable function satisfying on $a, b \in \mathcal{O}$, $|a +_{\mathcal{O}} b|_{\mathcal{O}} = |a|_{\mathcal{O}} + |b|_{\mathcal{O}}$.

It is easy to convert these rules into construction of partial computable function g. Proof is complete.

The similar result will be correct for Σ_a^{-1}-approximations of numberings.

3 Two Elements Families of Sets

Let S be the family of sets, $S = \{A, B\}$, where A is Σ_a^{-1}-set, B is Σ_b^{-1}-set, if $e(a) = 0$ and Π_b^{-1}-set, if $e(a) = 1$. Let R be computably enumerable set. Define a numbering ν^R:

$$\nu_n^R = B, n \in R$$
$$\nu_n^R = A, n \notin R.$$

Lemma 2. *A numbering ν^R is $\Sigma_{b+_{\mathcal{O}}a}^{-1}$-computable.*

Since A is Σ_a^{-1}, there is (due to definition and lemma) a fair parity Σ_a^{-1}-approximation $\langle \phi, \psi \rangle$, that define A. Since B is Σ_b^{-1} (or Π_b^{-1}), there is Σ_b^{-1}(or Π_b^{-1})- approximation $\langle \tau, \sigma \rangle$. To prove $\Sigma_{a+_{\mathcal{O}}b}^{-1}$-computability of ν we should construct a total computable function $f(n, x, s)$ and a partial computable function $g(n, x, s)$.

Firstly, we redefine function $\sigma(x, s)$:

We define $\sigma'(x, 0) = b$.

On the stage $s + 1$, if $\tau(x, s) \neq \tau(x, s + 1)$ then $\sigma'(x, s') = \sigma(x, s')$ for all $s' > s + 1$, otherwise we define $\sigma'(x, s) = b$.

New function σ will be always defined.

Since R is computable enumerable, there is uniformly computable sequence $\{R_s\}_{s\in\omega}$ of finite sets $R_0 \subseteq R_1 \subseteq R_2 \subseteq ...$ such that holds $R = \bigcup\limits_{s\in\omega} R_s$.

Now we can define f and g:

For all $x \in \omega$

s=0 $f(n, x, 0) = \phi(x, 0)$ $g(n, x, 0) = b +_{\mathcal{O}} \psi(x, 0)$ for all n
s+1 If $n \in R_s$ than $f(n, x, s + 1) = \tau(x, s + 1)$ $g(n, x, s + 1) = \sigma(x, s + 1)$
If $n \notin R_s$ than $f(n, x, s + 1) = \phi(x, s + 1)$ $g(n, x, s + 1) = b +_{\mathcal{O}} \psi(x, 0)$

Verification

1. $g(n, x, s) <_{\mathcal{O}} b +_{\mathcal{O}} a$. This holds, since $\psi(x, s + 1) <_{\mathcal{O}} a$.
2. $g(n, x, s) \downarrow \rightarrow g(n, x, s + 1) \downarrow$. It works, since function σ is always defined.
3. $f(n, x, s) \neq f(n, x, s + 1) \rightarrow g(n, x, s + 1) \downarrow \neq g(n, x, s)$. Here we have only one "dangerous" moment: if at some stage s we change enumeration from set A to set B, $\psi(x, s) = 0$ and $\sigma(x, s + 1) = b$. Then we have two cases:
   – if $e(a) = 1$ then $\phi(x, s) = 1$ (since $\langle \phi, \psi \rangle$ is fair parity) and $\tau(x, s + 1) = 1$ (since B is Π_b^{-1}) and there are no changes
   – if $e(a) = 0$ then $\phi(x, s) = 0$ (since $\langle \phi, \psi \rangle$ is fair parity) and $\tau(x, s + 1) = 0$ (since B is Σ_b^{-1}) and there are also no changes.

Lemma is complete.

It is easy to see, that for any computable enumerable sets R, Q

$$R \leq_m Q \Leftrightarrow \nu^R \leq \nu^Q$$

And since there are infinitely many c.e. m-degrees, we have

Theorem 1. *Let S be the family of sets, $S = \{A, B\}$, where A is Σ_a^{-1}-set, B is Σ_b^{-1}-set, if $e(a) = 0$ and Π_b^{-1}-set, if $e(a) = 1$. Then there are infinitely many nonequivalent $\Sigma_{b+_{\mathcal{O}}a}^{-1}$-computable numberings of family S.*

Corollary 1. *Any family $S = \{A, B\}$ of Σ_a^{-1}-sets has infinitely many nonequivalent Σ_c^{-1}-computable numberings, where $c = a +_{\mathcal{O}} a$ if $e(a) = 0$ and $c = 2^{a+_{\mathcal{O}}a}$ if $e(a) = 1$.*

For $e(a) = 0$ ν^R from Lemma 2 will be $a +_{\mathcal{O}} a$.

For $e(a) = 1$ ν^R will be $a +_{\mathcal{O}} a_1$, where $|a_1|_{\mathcal{O}}$ is successor of $|a|_{\mathcal{O}}$. It works, since $\Sigma_a^{-1} \subset \Sigma_{a_1}^{-1}$ and $e(a_1) = 0$. And $a +_{\mathcal{O}} c = 2^{a+_{\mathcal{O}}a}$.

4 Positive numbering

Goncharov, Lempp and Solomon in [1] have shown the existence of Σ_n^{-1}- computable Friedberg numbering of the family of all Σ_n^{-1}-sets has a , n is natural number. We use this construction with some minor modifications to show, that

Theorem 2. *Let S be Σ_a^{-1}-computable family of sets and there is $B \in S$ – Σ_b^{-1}-set if $e(a) = 0$(Π_b^{-1}-set if $e(a) = 1$), then S has a $\Sigma_{b+_{\mathcal{O}}a}^{-1}$-computable positive numbering.*

Since S is Σ_a^{-1}-computable, there is Σ_a^{-1}-computable numbering μ and let $\langle \phi, \psi \rangle$ be its fair parity approximation. Let $\langle \tau, \sigma \rangle$ be Σ_b^{-1}-(or Π_b^{-1}-) approximation of set B. We construct a $\Sigma_{b+_{\mathcal{O}}a}^{-1}$-computable numbering ν by giving its approximation $\langle f, g \rangle$ and $\emptyset'$-partial computable function h(approximated by uniformly partial computable functions h_s in the sense that $h(n) \downarrow = m$ if $h_s(n) = m$ for cofinitely many s, and $h(n)$ is undefined otherwise). We meet the following

Reqirements:

1. If $\mu_n = \mu_n'$ for some $n' < n$ then $h(n)$ is undefined.
2. If $\mu_n \neq \mu_n'$ for all $n' < n$ then either $h(n)$ is defined and $\mu_n = \nu_{h(n)}$; or μ_n is equal to set B, and there is $m \in \omega - range(h)$ such that $\mu_n = \nu_m$.
3. For any $m \notin range(h)$, $\nu_m = B$.

Construction.

Stage $s = 0$:

$f(n, x, 0) = 0, g(n, x, 0) \uparrow$ for all $n, x \in \omega$.

$h(0) = h_0(0) = 0$, and $h_0(n) \uparrow$ for all $n > 0$.

$M_0 = \emptyset$

Stage $s + 1$:

s+1.1 If $h_s(n)$ is defined and for some $n' < n$

$$\phi(n', x, s) = \phi(n, x, s) \text{ for all } x \in [0, h_s(n) + 1],$$

then let $h_{s+1}(n)$ be undefined.

s+1.2 If $h_s(n)$ is defined, $n > 0$ and for some $s' < s$ and $m \in range(h_{s'}) \setminus range(h_s)$ there is

$$f(m, x, s) = f(h_s(n), x, s) \text{ for all } x \in [0, h_s(n) + 1],$$

then let $h_{s+1}(n)$ be undefined.

s+1.3 If $h_s(n)$ is defined but h_{s+1} is undefined, then for each such n(in increasing order of n), set
$f(h_s(n), x, s') = \tau(x, s'), g(h_s(n), x, s') = \sigma(x, s')$ for all $s' > s$ and $x \in \omega$
$M_{s+1} = M_s \bigcup \{h_s(n)\}$

s+1.4 If $h_s(n)$ is undefined for $n \leq s$, then for all such n (in increasing order of n) let $h_{s+1}(n)$ be the least m not in $\bigcup\limits_{s' \leq s} range(h_s)$ and not equal to $h_{s+1}(n')$ for some $n' < n$.

s+1.5 If $h_{s+1}(n)$ is defined then let $f(h_{s+1}(n), x, s+1) = \phi(n, x, s+1)$, and $g(h_{s+1}(n), x, s+1) = b +_{\mathcal{O}} \psi(n, x, s+1)$ for all $x \in \omega$.

s+1.6 If $h_{s+1}(n)$ is defined then let $h_{s+2}(n) = h_{s+1}(n)$

Verification

1. If $\mu_n = \mu_{n'}$ for some $n' < n$, then $h_s(n)$ is undefined for infinitely many s by s+1.1.
2. If $\mu_n \neq \mu_{n'}$ for all $n' < n$, then $h(n)$ becomes undefined at most finitely often. If $h(n)$ becomes undefined by step s+1.2 infinitely often then $\mu_n = B$ and $\mu_n = \nu_m$ for some m.
3. This goes from s+1.4.

The same thoughts as in Theorem 1 prove, that ν is $\Sigma^{-1}_{b+_{\mathcal{O}}a}$-computable numbering. Now we show, that ν is positive.

Let $M = \bigcup\limits_s M_s$. M is the set of numbers n, where $\nu_n = B$ (it can be only one such n not from M). The set M is computable enumerable and

$$\nu_x = \nu_y \Leftrightarrow (x = y) \bigvee (x \in M \& y \in M).$$

It is easy to see, that a set $\{\langle x, y\rangle | \nu_x = \nu_y\}$ is computable enumerable.

Theorem is complete.

The same corollaries as in Theorem 1 hold here.

5 Infinite Roger semilattices

Here we make some connections between Theorems 1 and 2. Let S be Σ^{-1}_a-computable family and there are three different sets $A, B, C \in S$, A is Σ^{-1}_a-set and B, C are Σ^{-1}_b-sets if $e(a) = 0$(Π^{-1}_b-sets if $e(a) = 1$). According to Theorem 2, we have $\Sigma^{-1}_{b+_o a}$-computable positive numbering ν of family S. Without loss of generality, we assume that $\nu_0 = A$, $\nu_1 = B$ and C is used like the set B from Theorem 2. We define numberings μ and μ^R:

$$\mu_n = \nu_{n+2}$$
$$\mu_n^R = B, n \in R$$
$$\mu_n^R = A, n \notin R,$$

where R – some computable enumerable set.

It is easy to see, that

$$\mu^R \oplus \mu \leq \mu^Q \oplus \mu \Leftrightarrow \mu^R \leq \mu^Q$$

And $\mu^R \oplus \mu$ will be $\Sigma^{-1}_{b+_o a}$-computable numbering of family S. Due to Theorem 1 we have

Theorem 3. *Let S be Σ^{-1}_a-computable family and there are three different sets $A, B, C \in S$, A is Σ^{-1}_a-set and B, C are Σ^{-1}_b-sets if $e(a) = 0$(Π^{-1}_b-sets if $e(a) = 1$). Then there are infinitely many nonequivalent $\Sigma^{-1}_{b+_\mathcal{O} a}$-computable numberings of family S.*

Corollary 2. *For any Σ^{-1}_a-computable family S, $|S| > 2$, there are infinitely many Σ^{-1}_c-computable nonequivalent numberings of family S, where $c = a +_\mathcal{O} a$ if $e(a) = 0$ and $c = 2^{a+_\mathcal{O} a}$ if $e(a) = 1$.*

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